A class of homogeneous scalar-tensor cosmologies with a radiation fluid

Stoytcho S. Yazadjiev *

Department of Theoretical Physics, Faculty of Physics, Sofia University,

5 James Bourchier Boulevard, Sofia 1164, Bulgaria

Abstract

We present a new class of exact homogeneous cosmological solutions with a radiation fluid for all scalar-tensor theories. The solutions belong to Bianchi type VI_h cosmologies. Explicit examples of nonsingular homogeneous scalar-tensor cosmologies are also given.

In recent years, scalar-tensor theories attracted much attention in many areas of gravitational physics and cosmology. From theoretical point of view, it should be noted that these theories arise naturally form the low energy limit of string theory [1],[2]. In scalar-tensor theories, gravity is mediated not only by the space-time metric but also by a scalar field. The progress in the understanding of scalar-tensor theories is closely connected with finding of exact solutions. The aim of the present work is to present a new class of exact homogeneous and anisotropic scalar-tensor cosmologies with a radiation fluid. The found solutions belong to Bianchi type VI_h and, therefore, are different form previously known in the literature homogeneous solutions which belong to either Bianchi I, V, IX or Kantowski-Sachs cosmologies [3]-[7].

Scalar-tensor theories are described by the following action in the Jordan frame [8]-[10]:

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-\tilde{g}}\left(F(\Phi)\tilde{R} - Z(\Phi)\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \right.$$
$$\left. -2U(\Phi)\right) + S_m\left[\Psi_m;\tilde{g}_{\mu\nu}\right]. \tag{1}$$

Here, G_* is the bare gravitational constant, $\tilde{R}$ is the Ricci scalar curvature with respect to the space-time metric $\tilde{g}_{\mu\nu}$. The dynamics of the scalar field Φ depends on the functions $F(\Phi)$, $Z(\Phi)$ and $U(\Phi)$. In order for the gravitons to carry positive energy the function $F(\Phi)$ must be positive. The nonnegativity of the energy of the dilaton field requires that $2F(\Phi)Z(\Phi) + 3[dF(\Phi)/d\Phi]^2 \geq 0$. The action of matter depends on the material fields Ψ_m and the space-time metric $\tilde{g}_{\mu\nu}$ but does not involve the scalar field Φ in order for the weak equivalence principle to be satisfied. It should be mentioned

*E-mail: yazad@phys.uni-sofia.bg

that the most used parametrization in the literature is Brans-Dicke one, corresponding to $F(\Phi) = \Phi$ and $Z(\Phi) = \omega(\Phi)/\Phi$.

It is much clearer to analyze the equations in the so-called Einstein frame. Let us introduce the new variables $g_{\mu\nu}$ and φ, and define

$$\begin{aligned} g_{\mu\nu} &= F(\Phi)\tilde{g}_{\mu\nu} \ , \\ \left(\frac{d\varphi}{d\Phi}\right)^2 &= \frac{3}{4}\left(\frac{d\ln(F(\Phi))}{d\Phi}\right)^2 + \frac{Z(\Phi)}{2F(\Phi)} \ , \\ \mathcal{A}(\varphi) &= F^{-1/2}(\Phi) \ , \\ 2V(\varphi) &= U(\Phi)F^{-2}(\Phi) \ . \end{aligned} \tag{2}$$

From now on we will refer to $g_{\mu\nu}$ and φ as Einstein frame metric and dilaton field, respectively.

In the Einstein frame the action (1) takes the form

$$\begin{aligned} S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 4V(\varphi)\right) \\ + S_m[\Psi_m; \mathcal{A}^2(\varphi)g_{\mu\nu}] \end{aligned} \tag{3}$$

where R is the Ricci scalar curvature with respect to the Einstein frame metric $g_{\mu\nu}$. The Einstein frame field equations then are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_* T_{\mu\nu} + 2\partial_\mu\varphi\partial_\nu\varphi \\ - g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - 2V(\varphi)g_{\mu\nu} \ , \end{aligned}$$

$$\nabla^\mu\nabla_\mu\varphi = -4\pi G_*\alpha(\varphi)T + \frac{dV(\varphi)}{d\varphi} \ , \tag{4}$$

$$\nabla_\mu T^\mu_\nu = \alpha(\varphi)T\partial_\nu\varphi \ .$$

Here $\alpha(\varphi) = d\ln(\mathcal{A}(\varphi))/d\varphi$ and the Einstein frame energy-momentum tensor $T_{\mu\nu}$ is related to the Jordan frame one $\tilde{T}_{\mu\nu}$ via $T_{\mu\nu} = \mathcal{A}^2(\varphi)\tilde{T}_{\mu\nu}$. In the case of a perfect fluid one has

$$\begin{aligned} \rho &= \mathcal{A}^4(\varphi)\tilde{\rho}, \\ p &= \mathcal{A}^4(\varphi)\tilde{p}, \\ u_\mu &= \mathcal{A}^{-1}(\varphi)\tilde{u}_\mu. \end{aligned} \tag{5}$$

In the present work we consider the case $V(\varphi) = 0$.

We were able to find the following class of exact solutions of (4) for a radiation fluid with an equation of state $\rho = 3p$ ($\tilde{\rho} = 3\tilde{p}$):

$$
\begin{aligned}
ds^2 &= \sinh^4(at)\left(-dt^2 + dx^2\right) \\
&+ e^{nax/\sqrt{3}}\left(\sinh^{1+n}(at)e^{\sqrt{3}ax}dy^2 + \sinh^{1-n}(at)e^{-\sqrt{3}ax}dz^2\right), \\
8\pi G_* p &= \frac{1}{2}\left(1 - \frac{n^2}{3}\right)\frac{a^2}{\sinh^4(at)}, \\
\varphi(t) &= \varphi_0 + \frac{\lambda}{a}\ln|\tanh(at/2)|, \\
u_\mu &= -\sinh^2(at)\delta^0_\mu.
\end{aligned}
\tag{6}
$$

Here, n is given by $n^2 = 9 - 4\frac{\lambda^2}{a^2}$. The solutions depend on three parameters φ_0, a and λ where φ_0 is arbitrary, $a > 0$ and $\frac{3}{2} \leq \frac{\lambda^2}{a^2} \leq \frac{9}{4}$. The range of the coordinates is

$$
0 < t < \infty, \;\; -\infty < x, y, z < \infty. \tag{7}
$$

The presented exact homogeneous cosmologies are solutions of the Einstein frame equations (4) and, therefore, are solutions to all scalar-tensor theories. The metric has three dimensional nonabelian group of isometries G_3 [11] acting on three-dimensional orbits and belonging to Bianchi type VI_h with $h = -n^2/9$. The corresponding Jordan frame solutions can be simply obtained by using transformations (2) and (5).

Let us consider some properties of the solutions in the Einstein frame. There is evidently a big-bang singularity at $t = 0$ where the curvature invariants, energy density, pressure and the dilaton field diverge.

The expansion always takes place along the x-axis whereas the time evolution of the remaining axises depends on n. When $|n| < 1$, both axises expand. For $|n| = 1$ one of the axises expand while the other remains static. In the last case when $1 < |n| \leq \sqrt{3}$ one of the axises expands whereas the other is in contraction.

Three types of singular behaviour can be distinguished. For $|n| < 1$ we have anisotropic point like singularity - all three spacial directions shrink as the initial time $t = 0$ is approached. For $|n| = 1$ the singularity is of finite line type. This type of singular behaviour describes the contraction of two spacial directions whereas the third spacial direction neither contracts or expands with time. The infinite line singular behaviour occurs for $1 < |n| \leq \sqrt{3}$. In this case, one of the spacial directions expands when the initial time $t = 0$ is approached. In fact, near the singularity the dilaton dominates over the fluid and our model approaches a Bianchi type I model with a minimally coupled scalar field. Near the singularity the spacetime metric has a Kasner-like form

$$
g_{\mu\nu} \sim diag(-1, \eta^{2\gamma_1}, \eta^{2\gamma_2}, \eta^{2\gamma_3}) \tag{8}
$$

where η is the proper time and

$$\gamma_1 = \frac{2}{3}, \;\; \gamma_2 = \frac{1+n}{6}, \;\; \gamma_3 = \frac{1-n}{6}. \tag{9}$$

The Kasner exponents, as it should be expected, satisfy the Belinskii-Khalatnikov relations

$$\gamma_1 + \gamma_2 + \gamma_3 = 1, \tag{10}$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 - \frac{2}{9}\frac{\lambda^2}{a^2}. \tag{11}$$

With regard to the late evolution, the dilaton field rapidly goes to a constant value as t increases - in other words general relativistic behaviour is recovered at late times in both Einstein and Jordan frame.

In general, the properties of the solutions in the Jordan frame will depend on the particular scalar-tensor theory (i.e. on the particular function $\mathcal{A}(\varphi)$). For some scalar-tensor theories, Jordan frame solutions can exhibit a drastically different behaviour from that in the Einstein frame. In order to demonstrate that we shall consider the following scalar-tensor theory described by the functions $F(\Phi) = \Phi$ and $Z(\Phi) = \left(1 - 3\kappa^2 + 3\kappa^2\Phi\right)/2\kappa^2\Phi\left(1-\Phi\right)$ which correspond to $\mathcal{A}(\varphi) = \cosh(\kappa\varphi)$ where $\kappa > 0$.

Let us consider the Jordan frame solution with $n = 0$ (i.e Bianchi VI_0 type) which is given by:

$$\begin{aligned}
\Phi(t) &= \frac{4\tanh^{3\kappa}(at/2)}{\left[1+\tanh^{3\kappa}(at/2)\right]^2}, \\
d\tilde{s}^2 &= \Phi^{-1}(t)\sinh^4(at)\left(-dt^2 + dx^2\right) \\
&+ \Phi^{-1}(t)\sinh(at)\left(e^{\sqrt{3}ax}dy^2 + e^{-\sqrt{3}ax}dz^2\right), \\
8\pi G_*\tilde{p} &= 8\pi G_* p\Phi^2(t) = \frac{8a^2}{\sinh^4(at)}\frac{\tanh^{6\kappa}(at/2)}{\left[1+\tanh^{3\kappa}(at/2)\right]^4}, \\
\tilde{u}^\mu &= \Phi^{1/2}(t)u^\mu = \frac{2\tanh^{3\kappa/2}(at/2)}{\left[1+\tanh^{3\kappa}(at/2)\right]}\sinh^{-2}(at)\delta_0^\mu.
\end{aligned} \tag{12}$$

The curvature invariants $\tilde{I}_1 = \tilde{C}_{\mu\nu\alpha\beta}\tilde{C}^{\mu\nu\alpha\beta}$, $\tilde{I}_2 = \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$ and $\tilde{I}_3 = \tilde{R}^2$ for the space-time metric (12) are:

$$\tilde{I}_1 = 3a^4\left[\frac{\Phi(t)}{\sinh^6(t)}\right]^2\left[\left(1-\sinh^2(at)\right)^2 - 9\cosh^2(at)\sinh^2(at)\right], \tag{13}$$

$$\tilde{I}_2 = 9\left[\frac{\Phi(t)}{2\sinh^4(at)}\right]^2\left[\partial_t^2\ln(\Phi^{-1}(t)\sinh^2(at)) - a\coth(at)\partial_t\ln(\Phi^{-1}(t)\sinh(at))\right]^2$$

$$+\left[\frac{\Phi(t)}{2\sinh^4(at)}\right]^2\Big[\partial_t^2\ln(\Phi^{-1}(t)\sinh^4(at))$$
$$+\partial_t\ln(\Phi^{-1}(t)\sinh^4(at))\partial_t\ln(\Phi^{-1}(t)\sinh(at))-3a^2\Big]^2$$
$$+2\left[\frac{\Phi(t)}{2\sinh^4(at)}\right]^2\left[\partial_t^2\ln(\Phi^{-1}(t)\sinh(at))+\left(\partial_t\ln(\Phi^{-1}(t)\sinh(at))\right)^2\right]^2,$$

$$\tilde{I}_3=\left[\frac{\Phi(t)}{2\sinh^4(at)}\right]^2\Big[2\partial_t^2\ln(\Phi^{-1}(t)\sinh^4(at))+4\partial_t^2\ln(\Phi^{-1}(t)\sinh(at))+$$
$$3\left(\partial_t\ln(\Phi^{-1}(t)\sinh(at))\right)^2-3a^2\Big]^2.$$

All curvature invariants, the energy density and the pressure are regular everywhere for $\kappa\geq 2$ as can be seen. Therefore the solution has no big-bang singularity or any other curvature singularity.

Moreover, it can be shown that the solution under consideration is also geodesically complete for $\kappa\geq 2$. Since the space-time possesses group of isometries G_3, there are three conserved quantities along the Killing fields which generate G_3:

$$\begin{aligned}\Phi^{-1}(t)\sinh(at)e^{\sqrt{3}ax}\frac{dy}{d\tau}&=L_1,\\ \Phi^{-1}(t)\sinh(at)e^{-\sqrt{3}ax}\frac{dz}{d\tau}&=L_2,\\ \Phi^{-1}(t)\sinh^4(at)\frac{dx}{d\tau}-\frac{\sqrt{3}}{2}aL_1y+\frac{\sqrt{3}}{2}aL_2z&=L_3,\end{aligned}\tag{14}$$

where τ is the affine parameter.

Taking into account these conserved quantities, the geodesic equations can be reduced to

$$\begin{aligned}\frac{dt}{d\tau}&=\pm\left[\frac{\Phi(t)}{\sinh^4(at)}\right]^{1/2}M^{1/2}(t,x,y,z),\\ \frac{dy}{d\tau}&=L_1\frac{\Phi(t)}{\sinh(at)}e^{-\sqrt{3}ax},\\ \frac{dz}{d\tau}&=L_2\frac{\Phi(t)}{\sinh(at)}e^{\sqrt{3}ax},\\ \frac{dx}{d\tau}&=\frac{\Phi(t)}{\sinh^4(at)}\left[L_3+\frac{\sqrt{3}}{2}aL_1y-\frac{\sqrt{3}}{2}aL_2z\right],\end{aligned}\tag{15}$$

where

$$\begin{aligned}M\quad&(t,x,y,z)=\epsilon+L_1^2\frac{\Phi(t)}{\sinh(at)}e^{-\sqrt{3}ax}+L_2^2\frac{\Phi(t)}{\sinh(at)}e^{\sqrt{3}ax}\\ +\quad&\frac{\Phi(t)}{\sinh^4(at)}\left(L_3+\frac{\sqrt{3}}{2}aL_1y-\frac{\sqrt{3}}{2}aL_2z\right)^2,\end{aligned}\tag{16}$$

$\epsilon = 0$ and $\epsilon = 1$ for null and timelike geodesics, respectively.

Let us first consider the geodesics with $\epsilon = 1$ and $L_1 = L_2 = L_3 = 0$. From the explicit form of $\Phi(t)$ one sees that $\Phi(t)/\sinh^4(at)$ is a bounded function and we obtain

$$|\frac{dt}{d\tau}| = \left[\frac{\Phi(t)}{\sinh^4(at)}\right]^{1/2} < const. \tag{17}$$

Therefore, the only danger is if $t(\tau)$ could reach the value $t = 0$ for finite value of the affine parameter [12]. However, this can not occur since the integral

$$\int_0^t \left[\frac{\Phi(t)}{\sinh^4(at)}\right]^{-1/2} dt \tag{18}$$

is divergent for $\kappa \geq 2$. So, we conclude that the geodesics under consideration are complete.

The same reasons show that the geodesics with $L_3 \neq 0$, $L_1 = L_2 = 0$ and $\epsilon = 0, 1$ are complete,too.

Let us consider now the geodesics for which at least one of the constants L_1 or L_2 is different from zero ($\epsilon = 0, 1$ and L_3 arbitrary). In this case the derivatives $dt/d\tau$, $dx/d\tau$, $dy/d\tau$ and $dz/d\tau$ could become singular for finite values of the affine parameter only for $x(\tau)$. More precisely, the derivatives can turn singular for finite values of the affine parameter only if $x(\tau)$ could grow unboundedly to $+\infty$ when $L_2 \neq 0$ and to $-\infty$ when $L_1 \neq 0$. However, when $L_2 \neq 0$, $x(\tau)$ cannot grow unboundedly to $+\infty$ for finite values of τ since the derivative $dx/d\tau$ becomes negative for large $x(\tau)$ (large $L_2 z(\tau)$). In the same way, when $L_1 \neq 0$, $x(\tau)$ cannot grow unboundedly to $-\infty$ for finite values of τ since the derivative $dx/d\tau$ becomes positive for large $|x(\tau)|$ (large $L_1 y(\tau)$). We conclude thus that all derivatives are finite for finite values of the affine parameter. The only danger, therefore, is if $t(\tau)$ could reach the value $t = 0$ for finite value of τ. This cannot occur since the integral (18) is divergent and $M(t(\tau), x(\tau), y(\tau), z(\tau))$ is finite for finite values of τ.

So, we proved that all causal geodesics are complete. This result is not, of course, in contradiction with the famous singularity theorems. By computing the components of the Ricci tensor we have

$$\begin{aligned}\tilde{R}_{00} = \frac{3}{2}\Big[& a\coth(at)\partial_t \ln\left(\Phi^{-1}(t)\sinh(at)\right) \\ & -\partial_t^2 \ln\left(\Phi^{-1}(t)\sinh^2(at)\right)\Big] .\end{aligned} \tag{19}$$

We may see that the sign of $\tilde{R}_{00}$ changes. The strong energy condition is thus violated, and this explains why the singularity theorems can be evaded.

Summarizing, we have presented a new class of homogeneous and anisotropic cosmological solutions with radiation fluid for all scalar-tensor theories. The found solutions are of Bianchi type VI_h. The behaviour of the solutions in the Einstein frame has been described. The behaviour of the solutions in the physical Jordan frame depends on the particular scalar-tensor theory and can be very different from that in the

Einstein frame. As an illustration of that, explicit examples of geodesically complete homogeneous scalar-tensor cosmologies have been given.

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[12] It is instructive to consider the following example. Let us take $\kappa = 4/3$. Then the function $\Phi(t)/\sinh^4(at)$ is bounded and $|dt/d\tau| < const$. However, the integral (18) is convergent for $\kappa = 4/3$ and the geodesic with $L_1 = L_2 = L_3 = 0$ and $\epsilon = 1$ reaches $t = 0$ for a finite value of the affine parameter and cannot be continued. The space-time is thus singular as the curvature invariants show for $\kappa = 4/3$.